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Olivier Dudas

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Abstract

We show that for an algebraic reductive group G, the partition of a double Schubert cell in the flag variety G/B defined by Deodhar, and coming from a Bialynicki-Birula decomposition, is not a stratification in general. We give a counterexample for a group of type B_n, where the closure of some specific cell of dimension $2n$ has a non-trivial intersection with a cell of dimension $3n-3$.

INTRODUCTION

Let G be an algebraic reductive group defined over an algebraically closed field k together with a fixed Borel subgroup B containing a maximal torus T of G. The Coxeter system corresponding to these data will be denoted by (W, S). More precisely, $W = N_G(T)/T$ and S is the set of non-trivial elements $s \in W$ such that BsB is of minimal dimension. The opposite Borel subgroup B^* will be defined as the conjugate of B by the longest element w_0 of W.

We will be concerned with a refinement of the Bruhat stratification of the flag variety G/B. Recall that under the action of B (resp. B^*), this variety decomposes into a disjoint union of orbits, each of them containing a unique element of W. Such an orbit will be denoted by $Bw \cdot B$ (resp. $B^*w \cdot B$) and referred as the Schubert cell (resp. the opposite Schubert cell) corresponding to w.

Given two elements of the Weyl group w and v, Deodhar has defined in [Deo] a partition of the double Schubert cell $Bw \cdot B \cap B^*v \cdot B$ into affine smooth locally closed subvarieties of the flag variety G/B. This decomposition is not unique in general and depends on a reduced expression of w. When such an expression is chosen, the decomposition has a combinatorial definition: the set of cells is parametrized by some subexpressions of w, the distinguished ones, and each cell is isomorphic to $k^n \times (k^\times)^m$ where n and m can be defined in terms of the associated subexpression (see [Deo, theorem 1.1]).

In the special case where w is a Coxeter element, Deodhar was able to describe the closure of a cell (see [Deo, section 4]), giving thus a complete description of the geometry of the double Schubert cell. This particular example, together with the recent work of Webster and Yakimov on a more general decomposition (see [WY] and [We]), lead to the following expectations:

(i) the closure of a cell is a union of cells;

(ii) there is a natural order on the set of cells related to the Bruhat order, such that the closure of a cell has a non trivial intersection with all the smaller cells for this order.

Unfortunately, these two assertions fail in general, and we give two examples showing that the situation is much more complicated (section 2.2 and 2.3). At the present time, we have no clue for what can be the closure of a cell.

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1 DOUBLE SCHUBERT CELLS AND DEODHAR DECOMPOSITION

We recall in this section the principal result of [Deo], using a different approach due to Morel (see [Mo, Section 3]) which relies on a general decomposition theorem, namely the Bialynicki-Birula decomposition, applied to Bott-Samelson varieties.

Let $w \in W$ be an element of the Weyl group of G. The Schubert variety X_w associated to w is the closure in G/B of the Schubert cell $Bw \cdot B$. This variety is not smooth in general, but Demazure has constructed in [Dem] a resolution of the singularities, called the Bott-Samelson resolution, which is a projective smooth variety over X_w. The construction is as follows: we fix a reduced expression $w = s_1 \cdots s_\ell$ of w and we define the Bott-Samelson variety to be

$$BS = P_{s_1} \times_B \cdots \times_B P_{s_\ell}/B$$

where $P_{s_i} = B \cup Bs_iB$ is the standard parabolic subgroup corresponding to the simple reflection s_i. It is thus defined as the quotient of $P_{s_1} \times \cdots \times P_{s_\ell}$ by the right action of B^ℓ given by $(p_1, \ldots, p_\ell) \cdot (b_1, \ldots, b_\ell) = (p_1b_1, b_1^{-1}p_2b_2, \ldots, p_{\ell-1}^{-1}p_\ell b_\ell)$. The homomorphism $\pi : BS \longrightarrow X_w$ which sends the class $[p_1, \ldots, p_\ell]$ in BS of an element $(p_1, \ldots, p_\ell) \in P_{s_1} \times \cdots \times P_{s_\ell}$ to the class of the product $p_1 \cdots p_\ell$ in G/B is called the **Bott-Samelson resolution**. It is a proper surjective morphism of varieties and it induces an isomorphism between $\pi^{-1}(Bw \cdot B)$ and $Bw \cdot B$.

Now the torus T acts naturally on BS by left multiplication on the first component, or equivalently by conjugation on each component, so that π becomes a T-equivariant morphism. There are finitely many fixed points for this action, represented by the classes of the elements of $\Gamma = \{1, s_1\} \times \cdots \times \{1, s_\ell\}$ in BS; such an element will be called a **subexpression** of w.

For a subexpression $\gamma = (\gamma_1, \ldots, \gamma_\ell) \in \Gamma$ of w, we denote by $\gamma^i = \gamma_1 \cdots \gamma_i$ the i-th partial subword and we define the following two sets:

$$I(\gamma) = \{i \in \{1, \ldots, \ell\} \mid \gamma_i = s_i\}$$

and

$$J(\gamma) = \{i \in \{1, \ldots, \ell\} \mid \gamma^i s_i < \gamma^i\}.$$

With these notations, Deodhar's decomposition theorem (see [Deo, Theorem 1.1 and Corollary 1.2]) can be stated as follows:

Theorem 1.1 (Deodhar, 84). *There exists a family $(D_\gamma)_{\gamma\in\Gamma}$ of disjoint smooth locally closed subvarieties of $Bw \cdot B$ such that:*

(i) *D_γ is non empty if and only if $J(\gamma) \subset I(\gamma)$;*

(ii) *if D_γ is non empty, then it is isomorphic to $k^{|I(\gamma)|-|J(\gamma)|} \times (k^\times)^{\ell-|I(\gamma)|}$ as a variety;*

(iii) *for all $v \in W$, the double Schubert cell has the following decomposition:*

$$Bw \cdot B \cap B^*v \cdot B = \coprod_{\gamma\in\Gamma_v} D_\gamma$$

where Γ_v is the subset of Γ consisting of all subexpressions γ such that $\gamma^\ell = v$.

Remark 1.2. In the first assertion, the condition for a cell D_γ to be non-empty, that is $J(\gamma) \subset I(\gamma)$, can be replaced by:

$$\forall\, i = 2, \ldots, \ell \qquad \gamma^{i-1} s_i < \gamma^{i-1} \implies \gamma_i = s_i.$$

A subexpression $\gamma \in \Gamma$ which satisfies this condition is called a **distinguished subexpression**. For example, if $G = \mathrm{SL}_3(k)$ and $w = w_0 = sts$, then there are seven distinguished subexpressions, the only one being not distinguished is $(s, 1, 1)$.

Sketch of proof: the Bott-Samelson variety is a smooth projective variety endowed with an action of the torus T. Let us consider the restriction of this action to $\mathbb{G}_m$ through a strictly dominant cocharacter $\chi : \mathbb{G}_m \longrightarrow T$. Since this action has a finite number of fixed points, namely the elements of Γ, there exists a Bialynicki-Birula decomposition of the variety BS into a disjoint union of affine spaces indexed by Γ (see [BB, Theorem 4.3])

$$BS = \coprod_{\gamma \in \Gamma} C^\gamma.$$

In [Hä], Härterich has explicitly computed the cells C^γ. To describe this computation, we need some more notations: Φ will be the root system corresponding to the pair (G, T) and Φ^+ (resp. Φ^-) the set of positive (resp. negative) roots defined by B (resp. B^*). For any root $\alpha \in \Phi$ we denote by U_α the corresponding one-parameter subgroup and we choose an isomorphism $u_\alpha : k \longrightarrow U_\alpha$. The simple roots associated to the simple reflections of the reduced expression $w = s_1 \cdots s_\ell$ will be denoted by $\alpha_1, \ldots, \alpha_\ell$. Finally, we consider the open immersion $a_\gamma : \mathbb{A}_\ell \longrightarrow BS$ defined by

$$a_\gamma(x_1, \ldots, x_\ell) = [u_{\gamma_1(-\alpha_1)}(x_1)\gamma_1, \ldots, u_{\gamma_\ell(-\alpha_\ell)}(x_\ell)\gamma_\ell].$$

Then one can easily check that $\pi^{-1}(Bw \cdot B) = \mathrm{Im}(a_{(s_1, \cdots, s_r)})$. Moreover, Härterich's computations (see [Hä, Section 1]) show that for any subexpression $\gamma \in \Gamma$, one has:

$$C^\gamma \;=\; a_\gamma\big(\{(x_1, \ldots, x_\ell) \in \mathbb{A}_\ell \mid x_i = 0 \;\text{ if }\; i \in J(\gamma)\}\big).$$

Taking the trace of this decomposition with $\pi^{-1}(Bw \cdot B)$, one obtains a decomposition of the variety $\pi^{-1}(Bw \cdot B)$. Furthermore, the restriction of π to this variety induces an isomorphism with $Bw \cdot B$, and thus gives a partition of $Bw \cdot B$ into disjoint cells:

$$\pi^{-1}(Bw \cdot B) \;=\; \coprod_{\gamma \in \Gamma} \pi^{-1}(Bw \cdot B) \cap C^\gamma \;\simeq\; \coprod_{\gamma \in \Gamma} Bw \cdot B \cap \pi(C^\gamma) \;=\; Bw \cdot B.$$

If we define D_γ to be the intersection $Bw \cdot B \cap \pi(C^\gamma)$, then it is explicitly given by:

$$D_\gamma \simeq \pi^{-1}(D_\gamma) \;=\; a_\gamma\big(\{(x_1, \ldots, x_\ell) \in \mathbb{A}_\ell \mid x_i = 0 \;\text{ if }\; i \in J(\gamma) \;\text{ and }\; x_i \neq 0 \;\text{ if }\; i \notin I(\gamma)\}\big).$$

This description, together with the inclusion $\pi(C^\gamma) \subset B^* \gamma^\ell \cdot B$, proves the three assertions of the theorem.

□

Example 1.3. In the case where $G = \mathrm{SL}_3(k)$, and $w = w_0 = sts$, one can easily describe the double Schubert cell $Bw \cdot B \cap B^* \cdot B$. It is isomorphic to $BwB \cap U^*$ by the map $u \mapsto uB$, where U^* denotes the unipotent radical of B^*. Besides, by Gauss reduction, the set $BwBw^{-1} = BB^*$ consists of all matrices whose principal minors are non-zero. Hence,

$$BwB \cap U^* \;=\; \left\{ \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ c & b & 1 \end{pmatrix} \;\middle|\; c \neq 0 \;\text{ and }\; ab - c \neq 0 \right\}.$$

Considering the alternative $a = 0$ or $a \neq 0$, one has $BwB \cap U^* \simeq (k^\times)^3 \cup k \times k^\times$, which is exactly the decomposition given by the two distinguished expressions $(1, 1, 1)$ and $(s, 1, s)$.

Notations 1.4. For a subexpression $\gamma \in \Gamma$, we define the sequence

$$\Phi(\gamma) \;=\; \big(\gamma^i(-\alpha_i) \;\big|\; i = 1, \ldots, \ell \text{ and } \gamma^i(\alpha_i) > 0\big).$$

Using Härterich's computation for the cell C^γ and the definition of π, one can see that each element of $\pi(C^\gamma) \subset B^*\gamma^\ell \cdot B$ has a representative in the unipotent radical U^* of B^* which can be written in the following form:

$$\prod_{\alpha \in \Phi(\gamma)} u_\alpha(x_\alpha) \qquad \text{with each } \; x_\alpha \in k,$$

the product being taken with respect to the order on $\Phi(\gamma)$. At the level of D_γ, some of the variables x_α must be non-zero (those corresponding to $\gamma^i(-\alpha_i)$ with $\gamma_i = 1$) but the expression becomes unique, and it will be referred as the **canonical expression** in U^* of an element of D_γ.

2 On the closure of Deodhar cells

This section is devoted to the two questions raised in the introduction. Before recalling them, we make the statements more precise. For $w = s_1 \ldots s_\ell$ a reduced expression of an element w of W, we have defined in the previous section a desingularization of the Schubert variety X_w. One can embed this variety into a product of flag varieties as follows: we define the morphism $\iota : BS \longrightarrow (G/B)^\ell$ by

$$\iota([p_1, p_2, \ldots, p_\ell]) = (p_1 B, p_1 p_2 B, \ldots, p_1 p_2 \cdots p_\ell B).$$

Note that π is the last component of this morphism. Let $\gamma \in \Gamma$ be a subexpression of w. As a direct consequence of the construction of C^γ, one has

$$\iota(C^\gamma) \;\subset\; \prod_{i=1}^{\ell} B^*\gamma^i \cdot B.$$

Since BS is projective, ι is a closed morphism, and hence it sends the closure of a cell C^γ in BS to the closure of $\iota(C^\gamma)$. Therefore, it is natural to consider a partial order on the set Γ coming from to the Bruhat order on W since it describes the closure relation for Schubert cells. For $\delta \in \Gamma$, we define

$$\delta \preceq \gamma \quad \Longleftrightarrow \quad \gamma^i \leq \delta^i \;\text{ for all }\; i = 1, \ldots, \ell.$$

Then, by construction: $\displaystyle \overline{C^\gamma} \;\subset\; \bigcup_{\delta \preceq \gamma} C^\delta \quad \text{and} \quad \overline{D_\gamma} \;\subset\; \bigcup_{\delta \preceq \gamma} D_\delta$

where $\overline{D_\gamma}$ denotes the closure of D_γ in the Schubert cell $Bw \cdot B$. Now with these notations, the questions raised in the introduction can be rewritten as:

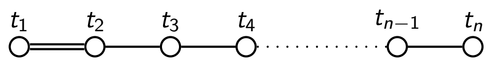
Question 2.1. *Is the closure of D_γ a union of cells ? In other terms, does the partition $(D_\gamma)_{\gamma \in \Gamma}$ define a stratification of the variety $Bw \cdot B$?*

Question 2.2. *For a subexpression $\delta \preceq \gamma$, do we have $\overline{D_\gamma} \cap D_\delta \neq \varnothing$?*

It is possible to give a positive answer to both of these questions in some specific cases -w a Coxeter element or γ maximal. However, this is not the case in general, and the situation can be even worse, as shown in the following sections.

II.1 - Chevalley formula in type B_n

From now on, G will be a quasi-simple group of type B_n, for example the orthogonal group $SO_{2n+1}(k)$. The Weyl group $W = W_n$ and its underlying root system correspond to the following Dynkin diagram:



The set of generators will be denoted by $S = \{t_1, \dots, t_n\}$ and the associated simple roots by $\{\beta_1, \dots, \beta_n\}$. There are n^2 positive roots, and their expression in terms of the simple ones is given by [Bou, Planche II]:

- $\alpha_i + \alpha_{i+1} + \cdots + \alpha_j$ for $1 \leq i \leq j \leq n$;
- $2\alpha_1 + \cdots + 2\alpha_i + \alpha_{i+1} + \cdots + \alpha_j$ for $1 \leq i < j \leq n$.

Recall that to each of these roots and their opposite correspond a one-parameter subgroup $u_\alpha : k \longrightarrow U_\alpha$. Since every element of a Deodhar cell can be written in terms of these subgroups (see notations 1.4), we need to recall the fundamental tool we will be using for all the computations, that is, the Chevalley commutator formula (see [Car, Theorem 5.2.2]). One may, and we will, choose indeed the family $(u_\alpha)_{\alpha\in\Phi}$ such that if $\alpha, \beta \in \Phi$ are any linearly independent roots and $x, y \in k$ any scalars, one has:

$$\big[u_\alpha(x)\,;u_\beta(y)\big] \;=\; u_\alpha(x)u_\beta(y)u_\alpha(-x)u_\beta(-y) \;=\; \prod_{i,j>0} u_{i\beta+j\alpha}(C_{ij\beta\alpha}\,(-y)^i x^j)$$

where the product is taken over all pairs of positive integers i, j for which $i\beta + j\alpha$ is a roots, in order of increasing $i+j$. For the simplicity of the proofs, we give here some explicit expressions of this formula in the specific cases we will encounter:

Formula 2.3. *Let $x, y \in k$. For $\alpha, \beta \in \Phi^-$ and $i = 2, \dots, n-1$, we have*

(i) *if $\alpha + \beta \notin \Phi$ then $u_\alpha(x)u_\beta(y)u_\alpha(-x) \;=\; u_\beta(y)$;*

(ii) *if $\alpha = -\beta_i$ and $\beta = -\beta_{i+1} - \cdots - \beta_n$ then $u_\alpha(x)u_\beta(y)u_\alpha(-x) \;=\; u_{\alpha+\beta}(\pm xy)u_\beta(y)$;*

(iii) *if $\alpha = -2\beta_1 - \beta_2 - \cdots - \beta_{n-1}$ and $\beta = -\beta_2 - \cdots - \beta_n$ then $u_\alpha(x)u_\beta(y)u_\alpha(-x) = u_{\alpha+\beta}(\pm xy)u_\beta(y)$;*

(iv) *if $\alpha = -\beta_i - \cdots - \beta_{n-1}$ and $\beta = -\beta_n$ then $u_\alpha(x)u_\beta(y)u_\alpha(-x) \;=\; u_\beta(y)u_{\alpha+\beta}(\pm xy)$;*

(v) *if $\alpha = -\beta_1 - \cdots - \beta_{n-1}$ and $\beta = -\beta_n$ then $\big[u_\alpha(x)\,;u_\beta(y)\big] \;=\; u_{2\alpha+\beta}(\pm x^2 y)u_{\alpha+\beta}(\pm xy)$.*

Remark 2.4. The values of the constants $C_{ij\beta\alpha}$ can be determined by [Car, Section 4.3]. Note that the signs of these constants depend on a choice on some of the elements of the Chevalley basis of the Lie algebra of G (namely, the extra-special pairs, see [Car, Section 4.2]). However, this will not be relevant in our computations and we will use the notation $\pm$.

II.2 - Obstruction to the stratification

In this section we give a negative answer to question 2.1. To do so, we consider an element w of W_n defined by the following reduced expression:

$$w = t_n t_{n-1} \cdots t_2 t_1 t_2 \cdots t_{n-1} t_n t_{n-1} \cdots t_2 t_1 t_2 \cdots t_{n-1}$$

and we define $\gamma, \delta \in \Gamma$ to be the following two distinguished subexpressions of w:

$$\gamma \;=\; (1, t_{n-1}, t_{n-2}, \dots, t_2, 1, t_2, \dots, t_{n-1}, 1, t_{n-1}, \dots, t_2, 1, t_2, \dots, t_{n-1})$$

and $$\delta \;=\; (1, t_{n-1}, t_{n-2}, \dots, t_2, t_1, 1, 1, \dots\dots\dots\dots\dots\dots\dots, 1, t_1, t_2, \dots, t_{n-1}).$$

The dimension of the cells associated to these subexpressions is given by theorem 1.1.(ii). One can easily check that $\dim D^\gamma = 2n$ and $\dim D^\delta = 3n - 3$ although the two subexpressions are related by $\delta \preceq \gamma$. Therefore, for $n \geq 4$, the closure of D^γ cannot contain the cell D^δ and in this situation, one can no longer give a positive answer to both of the questions. More precisely, we prove:

Proposition 2.5. *The closure of D^γ in the double Schubert cell $Bw \cdot B \cap B^* \cdot B$ contains a subvariety of D^δ of dimension n.*

Proof. (i) Let Ψ be the subset of the root system Φ defined by

$$\Psi = \{-2\beta_1 - \cdots - 2\beta_{n-1} - \beta_n\,;\, -\beta_2 - \cdots - \beta_n\,;\, -\beta_3 - \cdots - \beta_n\,;\, \ldots\,;\, -\beta_{n-1} - \beta_n\,;\, -\beta_n\}.$$

The sum of two elements of this subset is never a root, so that all the corresponding one-parameter subgroups commute. Associated to this set of roots, we define

$$V = \prod_{\beta\in\Psi} u_\beta(k^\times) \subset U^*.$$

By the previous remark and formula 2.3.(i), this product does not depend on any order on Ψ. In order to make the connection with the cells D_γ and D_δ, we define the corresponding variety in G/B by

$$\Omega = V\cdot B \subset B^*\cdot B.$$

It is an affine variety of dimension n, isomorphic to V. We show now that it is contained in both $\overline{D_\gamma}$ and D_δ, which will prove the assertion of the theorem.

(ii) Using [Bou, Section V.4.1], one can easily determine the elements of the sequence $\Phi(\delta)$; their opposite are given by

$$\begin{aligned} -\Phi(\delta) = \big(&\beta_n\,;2\beta_1+\beta_2+\cdots+\beta_{n-1}\,;\beta_2\,;\beta_3\,;\ldots;\beta_{n-2}\,;\beta_{n-1}+\beta_n\,;\beta_{n-2}\,;\ldots;\beta_3\,;\beta_2\,;\\ &2\beta_1+\beta_2+\cdots+\beta_{n-1}\,;\beta_1+\cdots+\beta_{n-1}\,;\beta_2+\cdots+\beta_{n-1}\,;\ldots;\beta_{n-1}\big).\end{aligned}$$

Recall from notations 1.4 that the elements of D_δ are parametrized by variables $(x_\beta)_{\beta\in\Phi(\delta)}$ living in $k^\times$ (whose for which $\delta_i = 1$) or k. For this specific subexpression, one can check that the first $(2n-2)$-th roots correspond to variables in $k^\times$ whereas the last $(n-1)$-th correspond to variables in k. Therefore, for $\mathbf{y} = (y_1,\ldots,y_n) \in (k^\times)^n$, we can consider the element of D_δ associated to the following specialization:

$$(x_\beta)_{\beta\in\Phi(\delta)} = (y_1, y_2, \ldots, y_{n-1}, y_n, -y_{n-1}, \ldots, -y_3, -y_2, 0, \ldots, 0).$$

The corresponding representative in U^* is thus given by

$$u_{\mathbf{y}} = u^*_{\beta_n}(y_1)u^*_{2\beta_1+\beta_2+\cdots+\beta_{n-1}}(y_2)\,\underbrace{u^*_{\beta_2}(y_3)\cdots u^*_{\beta_{n-1}+\beta_n}(y_n)\cdots u^*_{\beta_2}(-y_3)}_{v_{\mathbf{y}}}\,u^*_{2\beta_1+\beta_2+\cdots+\beta_{n-1}}(-y_2)$$

where, with a view of making the computations readable, we have denoted by $u^*_\alpha = u_{-\alpha}$ the one-parameter subgroup corresponding to the root $-\alpha$. By successive applications of formula 2.3.(i) and 2.3.(ii), the expression of $v_{\mathbf{y}}$ simplifies into

$$v_{\mathbf{y}} = u^*_{\beta_2+\cdots+\beta_n}(\pm y_3\cdots y_n)\cdots u^*_{\beta_{n-2}+\beta_{n-1}+\beta_n}(\pm y_{n-1}y_n)u^*_{\beta_{n-1}+\beta_n}(y_n).$$

Now, by formula 2.3.(i) and 2.3.(iii) we get

$$\begin{aligned} u_{\mathbf{y}} &= u^*_{\beta_n}(y_1)u^*_{2\beta_1+\cdots+2\beta_{n-1}+\beta_n}(\pm y_2\cdots y_n)\,v_{\mathbf{y}}\\ &= u^*_{\beta_n}(y_1)u^*_{2\beta_1+\cdots+2\beta_{n-1}+\beta_n}(\pm y_2\cdots y_n)u^*_{\beta_2+\cdots+\beta_n}(\pm y_3\cdots y_n)\cdots u^*_{\beta_{n-1}+\beta_n}(y_n).\end{aligned}$$

Since every element of V can be written in this form, this proves that D_δ contains the n-dimensional variety Ω.

(iii) As in (ii), it is easy to compute the sequence of roots occurring in the canonical expression in U^* of the elements of D_γ (see notations 1.4). Its opposite is given by

$$\begin{aligned} -\Phi(\gamma) = \big(&\beta_n\,;\beta_1+\cdots+\beta_{n-1}\,;\beta_2+\cdots+\beta_{n-1}\,;\ldots;\beta_{n-1}\,;\\ &\beta_n\,;\beta_1+\cdots+\beta_{n-1}\,;\beta_2+\cdots+\beta_{n-1}\,;\ldots;\beta_{n-1}\big).\end{aligned}$$

For $\mathbf{z} = (z_1,\cdots,z_n,t) \in (k^\times)^{n+1}$, let us consider the representative $u_{\mathbf{z}} \in U^*$ of the element of D_γ corresponding to the following choice of variables:

$$(x_\beta)_{\beta\in\Phi(\delta)} = (z_n, z_1t, z_2t^2, z_3t^2, \ldots, z_{n-1}t^2, t^{-2}, -z_1t, -z_2t^2, -z_3t^2, \ldots, -z_{n-1}t^2).$$

Because all the variables are non-zero, there is no need to check which root should correspond to a variable in $k^\times$ or k. Besides, we can apply formula 2.3.(i) to change the order of some terms in $u_{\mathbf{z}}$ and to get

$$u_{\mathbf{z}} = u^*_{\beta_n}(z_n)u^*_{\beta_1+\cdots+\beta_{n-1}}(z_1 t)\underbrace{\cdots u^*_{\beta_{n-1}}(z_{n-1}t^2)u^*_{\beta_n}(t^{-2})u^*_{\beta_{n-1}}(-z_{n-1}t^2)\cdots}_{v_{\mathbf{z}}} u^*_{\beta_1+\cdots+\beta_{n-1}}(-z_1 t).$$

Applying successively formula 2.3.(i) and 2.3.(iv) leads to the following expression for $v_{\mathbf{z}}$

$$v_{\mathbf{z}} = u^*_{\beta_n}(t^{-2})u^*_{\beta_2+\cdots+\beta_n}(\pm z_2)u^*_{\beta_3+\cdots+\beta_n}(\pm z_3)\cdots u^*_{\beta_{n-1}+\beta_n}(\pm z_{n-1}).$$

Then, by using formula 2.3.(i) and then 2.3.(v) we obtain

$$\begin{aligned} u_{\mathbf{z}} &= u^*_{\beta_n}(z_n)\left[u^*_{\beta_1+\cdots+\beta_{n-1}}(z_1 t)\,;\,u^*_{\beta_n}(t^{-2})\right] v_{\mathbf{z}} \\ &= u^*_{\beta_n}(z_n)u^*_{2\beta_1+\cdots+2\beta_{n-1}+\beta_n}(\pm z_1^2)u^*_{\beta_1+\cdots+\beta_n}(\pm z_1 t^{-1})v_{\mathbf{z}}. \end{aligned}$$

Finally, in this expression it is possible to evaluate the limit at $t=\infty$

$$\lim_{t\to\infty} u_{\mathbf{z}} = u^*_{\beta_n}(z_n)u^*_{2\beta_1+\cdots+2\beta_{n-1}+\beta_n}(\pm z_1^2)u^*_{\beta_2+\cdots+\beta_n}(\pm z_2)\cdots u^*_{\beta_{n-1}+\beta_n}(\pm z_{n-1}).$$

Once again, we observe that every element of V can be written in this form, which proves that $\Omega = V\cdot B$ is contained in $\overline{D_\gamma}$. ☐

Corollary 2.6. *For any positive integer n, there exist $w\in W$, a reduced expression of w, and $\gamma,\delta\in\Gamma_1$ two subexpressions of w such that:*

- $D_\delta \not\subseteq \overline{D_\gamma}$;
- $\dim \overline{D_\gamma}\cap D_\delta \geq n$.

In particular, this gives a negative answer to question 2.1.

II.3 - Disjointness of cells

We move now attention to the problem raised in question 2.2. We assume that $n=3$ and we consider the following two distinguished subexpressions of w_0 associated to the reduced expression $w_0 = t_3t_2t_1t_2t_3t_2t_1t_2t_1$

$$\sigma = (1, t_2, 1, t_2, 1, t_2, t_1, 1, t_1)$$

and

$$\tau = (1, t_2, t_1, 1, 1, t_2, 1, t_2, t_1).$$

We have $\tau \preceq \sigma$, and the corresponding cells are subvarieties of $B^*t_2\cdot B$ of dimension 6.

Proposition 2.7. *The closure $\overline{D_\sigma}$ of D_σ in the Schubert cell $Bw_0\cdot B$ is disjoint from the cell D_τ, giving hence a negative answer to question 2.2.*

Proof. Using [Bou, Section V.4.1], one can compute the one-parameter subgroups occurring in the canonical expression in U^* of the elements of D_σ and D_τ (see notations 1.4). They are associated to the following sequences of roots:

$$-\Phi(\sigma) = \big(\beta_3\,;\beta_1+\beta_2\,;\beta_2\,;\beta_3\,;2\beta_1+\beta_2\,;\beta_1+\beta_2\big)$$

and

$$-\Phi(\tau) = \big(\beta_3\,;2\beta_1+\beta_2\,;\beta_2+\beta_3\,;\beta_1\,;2\beta_1+\beta_2\,;\beta_1+\beta_2\big).$$

By definition, both of the cells D_σ and D_τ are contained in $B^*t_2\cdot B$, but since the simple negative root $-\beta_1$ does not occur in $\Phi(\sigma)$, the cell D_σ is actually contained in $(B^*\cap {}^{t_1}B^*)t_2\cdot B$, which is a closed subvariety of codimension 1 in $B^*t_2\cdot B$. Therefore, the closure of D_σ in the double Schubert cell $Bw_0\cdot B\cap B^*t_2\cdot B$ is also contained in $(B^*\cap {}^{t_1}B^*)t_2\cdot B$.

On the other hand, $-\beta_1$ occurs only once in $\Phi(\tau)$ and corresponds to a variable in $k^\times$: more precisely, if $i=7$ then

- $\tau^i = t_2t_1t_2$ and $\tau_i = 1$ so that $i\notin I(\tau)$ corresponds to a variable in $k^\times$;
- $\tau^i(-\alpha_i) = \tau^i(-\beta_1) = t_2t_1t_2(-\beta_1) = -\beta_1$

so that the cell D_τ is disjoint from $(B^*\cap {}^{t_1}B^*)t_2\cdot B$ and then from the closure of D_σ. ☐

Remark 2.8. This situation is not specific to the low-dimensional cells. One can actually extend this example to the type B_n for any $n \geq 3$ by considering the concatenation of σ and τ with the subexpression of $v = t_n \cdots t_2 t_1 t_2 \cdots t_n$ defined by

$$\eta = (1, 1, \ldots, 1, t_2, 1, t_2, 1, \ldots, 1).$$

The Deodhar cells corresponding to the subexpressions $\tilde{\sigma} = \eta \cdot \sigma$ and $\tilde{\tau} = \eta \cdot \tau$ are now of dimension $2n + 2$, and satisfy indeed the previous proposition.

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